Notes on Dilaton Quantum Cosmology[1]

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ABSTRACT: In these notes we address the canonical quantization of the cosmological models which appear as solutions of the low energy effective action of closed bosonic string theory (dilaton models). The analysis is restricted to the quantization of the mini-superspace models given by homogeneous and isotropic cosmological solutions. We study the different conceptual and technical problems arising in the Hamiltonian formulation of these models as a consequence of the presence of the so called Hamiltonian constraint. In particular we address the problem of time in quantum cosmology, the characterization of the symmetry under clock reversals arising from the existence of a Hamiltonian constraint, and the problem of imposing boundary conditions on the space of solutions of the Wheeler–DeWitt equation.

[1]Chapter 3 of the book *Trends in General Relativity and Quantum Cosmology*, Nova Science Publishers, N. Y. (2005).

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1 Introduction

String cosmology received considerable attention in the last decade because of the new scenario that it proposes for the early universe [52, 53, 23]. When the high energy modes of the strings become negligible, the dynamical evolution of the universe begins to be dominated by the massless fields which act as the matter source of gravitational dynamics. This phase of the universe is commonly called the *dilatonic era*. The purpose of the present notes is to provide a consistent quantum description for this epoch. We shall address the formal aspects of the problem, more precisely, the obtention of a wave function allowing for a clear definition of probability within the context of the minisuperspace approximation.

In the minisuperspace picture all except a few degrees of freedom of the gravitational and matter fields are frozen at the classical level, so that the problem to be solved reduces from quantum field theory to quantum mechanics. Most developments in quantum cosmology have been achieved within the minisuperspace approximation. However, though the reduction to a finite number of degrees of freedom considerably simplifies the problem of obtaining a consistent quantum cosmology, the fact that the dynamical classical theory includes the general covariance as a central feature is an obstruction to a straightforward application of standard quantum mechanics.

This feature is most apparent when the classical theory of the gravitational field (even with the inclusion of matter) is formulated in the Hamiltonian form, as one immediately obtains that the dynamical evolution is governed by a Hamiltonian $\mathcal{H}$ which vanishes on the trajectories of the system [29, 6, 46]:

$$\mathcal{H} = G^{ik} p_i p_k + V(q) = 0.$$

Thus the theory involves a constraint which is quadratic in the momenta, which reflects the reparametrization invariance of the corresponding action (see below), i.e., the fact that the separation between successive spatial hypesurfaces in spacetime is arbitrary. In general, one also obtains linear constraints, analogous to those of Yang–Mills theories [6]. These linear constraints assure the invariance of the theory under a change of the spatial

coordinates used to represent the spatial geometry of each hypersurface. We shall not discuss this point here, as their role is not central in the minisuperspace picture.

The quantization of constrained systems can be analyzed within both the canonical approach and the path integral formulation (see, for example, Ref. [31]). In the first of these frameworks, the direct application to cosmological models of the well known Dirac program [20] leads to the Wheeler–DeWitt equation [19], which is a second order equation in all the derivatives of the wave function Ψ. The solutions to this equation do not depend explicitly on the "time" parameter τ, but only on the coordinates, which reflects the reparametrization invariance of the classical theory. This reparametrization invariance means that the integration parameter τ is not a "true" time. This absence of a notion of time in the Wheeler–DeWitt quantization program is a serious obstacle for understanding the results in terms of conserved positive-definite probabilities [28, 7, 22]. Within dilaton cosmology this problem has received considerable attention (see, for example, Refs. [14, 13, 15, 16, 26]).

The canonical program admits an alternative approach to the Dirac method, which relies on the idea that it is necessary to "reduce" the theory before quantization, i.e., to find a "true" time at the classical level. If this reduction can be effectively performed, the original action can be reduced to an ordinary action without the reparametrization symmetry of the former. In that case the quantization program continues as if we were dealing with an ordinary classical theory. The theory can be then quantized by means of a Schrödinger equation with its typical conserved positive-definite probabilities.

Our aim will be then the analysis of these points by studying the cosmological models which appear as solutions of the low energy effective action of closed bosonic string theory [24, 23]. In Section 2 we shall begin by reviewing dilaton gravity, in particular homogeneous and isotropic cosmological solutions, whose formulation we shall put in Hamiltonian form. Then in Section 3 we shall address the problem of time and quantization, both in the usual Wheeler–DeWitt scheme as well as in the Schrödinger formulation, and we shall carefully discuss the relation between the corresponding solutions, their possible equivalence, and their role in selecting solutions with physical meaning (in particular when

an extrinsic time must be introduced). Section 4 will be devoted to a more conceptual than technical discussion, with a wider scope which goes beyond the particular problem of dilaton models. Finally, in Section 5 we briefly summarize the essential points of the whole discussion.

2 Bosonic string theory and cosmological models

2.1 General theory

The action describing the world-sheet dynamics of strings on a curved manifold in presence of background fields has the form [39, 38]

$$\begin{aligned} S_{WS} &= \frac{1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{h}\left(h_{\alpha\beta}g_{\mu\nu}(X)+i\varepsilon_{\alpha\beta}B_{\mu\nu}(X)\right)\partial^\alpha X^\mu\partial^\beta X^\nu \\ &+\frac{1}{2\pi}\int d\sigma d\tau\sqrt{h}R(X)\phi(X), \end{aligned} \tag{1}$$

where $h_{\alpha\beta}$ is the metric on the string world-sheet, R is the Ricci scalar related with this metric, $g_{\mu\nu}$ is the metric of the spacetime on which the theory is formulated, $B_{\mu\nu}$ is an antisymmetric field (commonly known as the NS-NS two-form field) and ϕ is the scalar dilaton field. These three fields appear in the massless spectrum of closed bosonic string theory. We have noted α, β for the indices corresponding to the coordinates on the two-dimensional world-sheet, while the indices μ, ν, ρ correspond to the D-dimensional spacetime coordinates. The parameter α' (commonly known as the Regge slope) is the inverse of the string tension, $T = 1/(2\pi\alpha')$, which defines the scale of the theory at the quantum level. Clearly, the action (1) defines a two-dimensional field theory; this theory is invariant under the transformations

$$\delta B_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu, \qquad \delta\phi = \phi_0, \tag{2}$$

with Λ_μ an arbitrary vector and ϕ_0 a constant.

The action (1) is invariant under the Weyl –conformal– transformation $h_{\alpha\beta} \to \Omega^2(\tau,\sigma)h_{\alpha\beta}$. If we require that this symmetry holds also at the quantum level (no conformal anomalies) the beta functions must vanish [38, 23]; then at first order in the α' power expansion and

introducing the strength tensor $H_{\mu\nu\rho}$ associated to the antisymmetric field $B_{\mu\nu}$

$$\mathbf{H}_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}, \tag{3}$$

we obtain:

$$\begin{aligned} R_{\mu\nu} + \nabla_\mu \nabla_\nu \phi - \frac{1}{4}\mathbf{H}_{\mu\rho\delta}\mathbf{H}^{\rho\delta}_{\nu} &= 0, \\ \nabla^\delta \mathbf{H}_{\delta\mu\nu} - \nabla^\delta \phi \mathbf{H}_{\delta\mu\nu} &= 0, \\ c - \nabla_\mu \nabla^\mu \phi + \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{6}\mathbf{H}_{\mu\nu\rho}\mathbf{H}^{\mu\nu\rho} &= 0, \end{aligned} \tag{4}$$

where, in principle, $c = 2(D-26)/(3\alpha')$. However, c can be changed by including more fields, so that in what follows we shall consider it as an arbitrary real number. Note then the difference between the theory for a point particle and the theory for a string: the quantum theory for the last one can not be consistently formulated in an arbitrary background, but, instead, it imposes restrictions on the admissible external fields.

Now, we are interested in a spacetime formulation of the theory of gravitation, analogous to the Einstein–Hilbert action that we have in General Relativity. It can be shown that the equations (4) can be interpreted as the Euler–Lagrange equations of motion of a field theory corresponding to the following effective action:

$$S_{SF} = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} e^{-\phi} \left(-c + R + \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{12}\mathbf{H}_{\mu\nu\rho}\mathbf{H}^{\mu\nu\rho} \right), \tag{5}$$

where G_N is the D-dimensional Newton constant and R is the Ricci scalar of the spacetime. Thus, we can understand this as the low energy effective action describing the large tension limit ($\alpha' \to 0$) of closed bosonic string theory. A consistent configuration of background fields for the formulation of string theory must be a classical solution obtained from the variational principle corresponding to the action (5). In brief, the requirement of preserving at the quantum level the conformal invariance of the two-dimensional world sheet theory, formulated up to the first order in the inverse of the string tension, leads to the same equations of motion resulting from the variational principle $\delta S = 0$ imposed on the D-dimensional field theory given by (5).

A more familiar formulation can be obtained by redefining the fields as $g_{\mu\nu} \to e^{\phi} g_{\mu\nu}$, so that the action of the spacetime theory becomes

$$\begin{aligned} S_{EF} &= \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} \\ &\times \left(R - c e^{2\phi/(D-2)} + \frac{1}{D-2} \nabla_\mu \phi \nabla^\mu \phi - \frac{e^{-4\phi/(D-2)}}{12} \mathbf{H}_{\mu\nu\rho} \mathbf{H}^{\mu\nu\rho} \right). \end{aligned} \tag{6}$$

Thus we have obtained the D-dimensional Einstein action including coupling terms with the dilaton and the antisymmetric field. This form for the effective field theory is known as *Einstein frame action*, while (5) is commonly called the *string frame action*. The interpretation of the gravitational aspects of the theory is more clear in the Einstein frame: in the particular case $D = 4$, the variational principle $\delta S = 0$ imposed to this new form of the effective action leads to the equations

$$\nabla_\mu \partial^\mu \phi + c e^{\phi} - \frac{1}{16} e^{-2\phi} \mathbf{H}^2 = 0, \tag{7}$$

$$\nabla_\delta \mathbf{H}^\delta_{\mu\nu} + 2 \nabla_\delta \phi \mathbf{H}^\delta_{\mu\nu} = 0, \tag{8}$$

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{c}{2} g_{\mu\nu} e^{\phi} &= \frac{1}{2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right) + \\ &+ \frac{1}{4} e^{-2\phi} \left(\mathbf{H}_{\mu\rho\delta} \mathbf{H}^{\rho\delta}_\nu - \frac{1}{6} g_{\mu\nu} \mathbf{H}^2 \right), \end{aligned} \tag{9}$$

and the Bianchi identities

$$\nabla_{[\mu} \mathbf{H}_{\mu\rho\delta]} = 0. \tag{10}$$

We can recognize in (9) the Einstein equations with a cosmological function given by $\Lambda = c e^{\phi}$ and with the energy-momentum tensor of the dilaton and the antisymmetric fields as the source. The relation between the string frame and the Einstein frame formulations can be clarified by considering a solution of the equations (5) in the case of homogeneity and isotropy. Because the metric ds^2 in the string frame is related to the corresponding metric $d\tilde{s}^2$ in the Einstein frame by $ds^2 = e^{\phi} d\tilde{s}^2$, one can easily translate the results. For example, for the case $c = 0$ a possible solution is a flat cosmology with a metric ds^2 where the scale factor behaves like $a \sim \tau^{1/3}$. By recalling the corresponding evolution of the dilaton with τ, this behavior is translated to the Einstein frame as an evolution of the

scale factor $b \sim \tau^{1/2}$, that is, to the same evolution of a radiation-dominated universe (see [27] for a detailed discussion).

2.2 Cosmological models and Hamiltonian formulation

Consider a cosmological model with a finite number of degrees of freedom identified by the coordinates q^i (geometrical and matter degrees of freedom). The Lagrangian form for the action of such a minisuperspace is

$$S[q^i, N] = \int_{\tau_1}^{\tau_2} N \left(\frac{1}{2N^2} G_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau} - V(q) \right) d\tau \tag{11}$$

where a spatial integration must be understood in the integrand. In (11), G_{ij} is the reduced version of the DeWitt supermetric (see, for example, [6]), V is the potential, which depends on the curvature and the coupling between the fields, and $N(\tau)$ is the lapse function determining the separation between spacelike hypersurfaces in spacetime [6, 29, 46].

The role of constraints and Lagrange multipliers becomes manifest in the Hamiltonian formulation, being this formalism the best suited for the canonical quantization of cosmological models. If we define the canonical momenta as

$$p_i = \frac{1}{N} G_{ij} \frac{dq^j}{d\tau},$$

we obtain the Hamiltonian form of the action

$$S[q^i, p_i, N] = \int_{\tau_1}^{\tau_2} \left(p_i \frac{dq^i}{d\tau} - N\mathcal{H} \right) d\tau, \tag{12}$$

where

$$\mathcal{H} = G^{ij}\, p_i p_j + V(q). \tag{13}$$

Under arbitrary changes of the coordinates q^i, the momenta p_i and the lapse function N we obtain

$$\delta S = p_i \delta q^i \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \left[\left(\frac{dq^i}{d\tau} - N \frac{\partial \mathcal{H}}{\partial p_i} \right) \delta p_i - \left(\frac{dp^i}{d\tau} + N \frac{\partial \mathcal{H}}{\partial q_i} \right) \delta q^i - \mathcal{H} \delta N \right] d\tau. \tag{14}$$

Then if we demand the action to be stationary when the coordinates q^i are fixed at the boundaries, we obtain on the classical path the Hamilton canonical equations

$$\frac{dq^i}{d\tau} = N[q^i, \mathcal{H}], \qquad \frac{dp_i}{d\tau} = N[p_i, \mathcal{H}] \tag{15}$$

and the Hamiltonian constraint

$$\mathcal{H} = 0. \tag{16}$$

Two features of the dynamics can thus be remarked. The first one is that the presence of the constraint $\mathcal{H} = 0$ restricts possible initial conditions to those lying on the constraint surface. The second is that the evolution of the lapse function N is arbitrary, i.e., it is not determined by the canonical equations; hence, the separation between two successive three-surfaces is arbitrary, which constitutes the minisuperspace version of the many-fingered nature of time of the full theory of gravitation [6, 46].

The field equations yielding from the spacetime action (5) admit homogeneous and isotropic solutions in four dimensions [1, 49, 50, 27]. Such solutions have a metric of the Friedmann–Robertson–Walker form:

$$ds^2 = N(\tau)d\tau^2 - a^2(\tau)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right), \tag{17}$$

where a is the scale factor and $k = (-1, 0, 1)$ determines the curvature. For the dilaton ϕ and the field strength $H_{\mu\nu\rho}$ the homogeneity and isotropy requirements lead to

$$\phi = \phi(\tau) \qquad \mathbf{H}_{ijk} = \lambda(\tau)\varepsilon_{ijk} \tag{18}$$

where ε_{ijk} is the volume form on the constant-time surfaces and λ is a real number. The Bianchi identities (10) imply that λ does not depend on the parameter τ. An important aspect of these cosmological solutions is that they allow for a conceptually new scenario for the early universe, where the standard big bang is replaced by a phase of finite curvature [24, 23].

Let us write down the explicit form of the action for the models to be considered here. If we define $b^2(\tau) \equiv e^{2\Omega(\tau)}$, the Lagrangian form of the Einstein frame action in four

dimensions for the case $\lambda = 0$, which corresponds to the two-form field $B_{\mu\nu}$ equal to zero, is given by

$$S = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau N e^{3\Omega}\left[-\frac{\dot{\Omega}^2}{N^2} + \frac{\dot{\phi}^2}{N^2} - 2ce^{\phi} + ke^{-2\Omega}\right], \tag{19}$$

where a dot stands for $d/d\tau$. On the other hand, in the case $k = 0$ (flat universe) we can write

$$S = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau N e^{3\Omega}\left[-\frac{\dot{\Omega}^2}{N^2} + \frac{\dot{\phi}^2}{N^2} - 2ce^{\phi} - \lambda^2 e^{-6\Omega-2\phi}\right]. \tag{20}$$

We have absorbed the factor $(8\pi G_N)^{-1}$ by redefining the fields.

The Hamiltonian form of the Einstein frame action for the models considered reads

$$S = \int_{\tau_1}^{\tau_2} d\tau \left[p_\Omega \dot{\Omega} + p_\phi \dot{\phi} - N\mathcal{H}\right]. \tag{21}$$

For $\lambda = 0$ the Hamiltonian constraint is

$$\mathcal{H} = \frac{1}{2}e^{-3\Omega}\left(-p_\Omega^2 + p_\phi^2 + 2ce^{6\Omega+\phi} - ke^{4\Omega}\right) = 0, \tag{22}$$

while for $k = 0$ we have

$$\mathcal{H} = \frac{1}{2}e^{-3\Omega}\left(-p_\Omega^2 + p_\phi^2 + 2ce^{6\Omega+\phi} + \lambda^2 e^{-2\phi}\right) = 0. \tag{23}$$

These two Hamiltonian constraints, or their scaled forms $H \equiv 2e^{3\Omega}\mathcal{H} = 0$ (see Appendix A), will be the starting point for the canonical quantization of the models.

3 Quantization

3.1 Schrödinger and Wheeler–DeWitt equations

The standard procedure for quantizing the minisuperspaces models described by the Hamiltonian constraints (22) and (23) is to turn them into operators and make them act on a wave function (Dirac method). This prescription yields the usual Wheeler–DeWitt equation, which is an hyperbolic equation of second order in all its derivatives. However, the absence of a true time in the classical formalism (which is reflected in the appearance of the Hamiltonian constraint $\mathcal{H} = 0$) makes difficult the interpretation of the resulting wave function in terms of a conserved positive-definite inner product. One

knows in fact how to define a conserved positive-definite inner product for the solutions of a Schrödinger equation, and not for the solutions of a Klein–Gordon type equation as the Wheeler–DeWitt equation. Other problems associated with the Wheeler–DeWitt equation are discussed in [40, 41, 42].

However, in certain cases it is possible, by means of the identification of a time variable among the canonical variables, to extract from the time independent Wheeler–DeWitt equation a time dependent Schrödinger equation. In this case a clear probability interpretation can be given to the space of solutions of the Schrödinger equation. As Barbour said [5], in canonical quantum gravity the situation is in a certain sense inverted with respect to ordinary quantum mechanics where the time independent Schrödinger equation $\widehat{h}\psi = E\psi$ is derived from the more general time dependent Schrödinger equation $\widehat{h}\psi = i\frac{\partial\psi}{\partial t}$. In canonical quantum gravity a time dependent Schrödinger equation has to be extracted from a time independent Schrödinger equation with autovalue zero $\widehat{\mathcal{H}}\psi = 0$ (Wheeler–DeWitt equation).

There are mainly two possibilities for defining a Schrödinger equation for a dynamical system with a constraint $\mathcal{H} = 0$ quadratic in all its momenta. The first possibility it to perform a canonical transformation which transforms the quadratic Hamiltonian constraint in a constraint linear in one of its canonical momenta. If a certain positivity condition is satisfied by this linear momentum, its canonically conjugated coordinate can then be defined as the "real" time parameter for the evolution of the system [7, 22, 9].

The other possibility is to factorize the scaled Hamiltonian constraint $H \equiv 2e^{3\Omega}\mathcal{H} = p_0^2 - h^2 = 0$ (with $h^2 = p_\mu p^\mu + V(q^\mu, q^0)$) in two disjoint sheets $H = (p_0 + h)(p_0 - h) = 0$, given by the two signs of the momentum p_0 conjugated to the coordinate q^0 that one wants to identify with time (modulo a sign). Below we shall analyze the conceptual meaning of this factorization. A time dependent Schrödinger equation can then be associated to each factor $p_0 \pm h$. The splitting of the original constraint in two constraints requires the existence of a non vanishing momentum p_0 (a *true* time must not invert its sense of evolution), which may not happen in the original variables describing the model. Indeed, for q^0 to be a right time choice, the condition $[q^0, \mathcal{H}] > 0$ must be fulfilled (see [28]

and the Appendix A), leading this condition to the requirement that $p_0 \neq 0$. Besides, for obtaining a proper quantum theory in each sheet, the operator corresponding to the reduced Hamiltonian h must be a self-adjoint operator in order to have a unitary evolution. Given that in the factorization process the reduced Hamiltonian h is obtained by taking a square root $\left(h = \sqrt{p_\mu p^\mu + V\left(q^\mu, q^0\right)}\right)$, the associated operator $\hat{h}$ will be self-adjoint only if the term under the square root is positive defined, being this fact an important constraint to the reduction process that we have described. It may also happen that the model can not be reduced in the original canonical variables, being then necessary to perform a canonical transformation in order to properly reduce the system with a self-adjoint Hamiltonian operator $\hat{h}$. In that case it is possible that the selected time variable depends both on the original canonical coordinates and momenta ($t \equiv q^0 = q^0\left(q^\mu, p_\mu\right)$), being these kind of time variables called *extrinsic times* [32, 54] (in contrast to the *intrinsic times* which depend only on the canonical coordinates q^μ).

If the reduction process can be successfully performed, then we will have a pair of Hilbert spaces, each one with its corresponding Schrödinger equation of first order in $\partial/\partial q^0$ with a reduced Hamiltonian operator $\hat{h}$. The physical inner product is defined in each space as

$$(\Psi_2|\Psi_1) = \int dq\, \delta(q^0 - \tilde{q}^0)\Psi_2^*\Psi_1. \tag{24}$$

We could say that the Schrödinger quantization preserves the topology of the constraint surface: the splitting of the classical solutions into two disjoint subsets has its quantum version in the splitting of the theory in two Hilbert spaces [10, 47].

3.2 Intrinsic time

We shall begin by considering a generic (scaled) Hamiltonian constraint of the form

$$-p_1^2 + p_2^2 + Ae^{(aq^1+bq^2)} = 0, \tag{25}$$

with $a \neq b$. This constraint admits an intrinsic time because the potential does not vanish for finite values of the coordinates (see Appendix A; for models not fulfilling this condition see the next section). This Hamiltonian constraint corresponds to several dilaton

cosmologies, namely, to the cases $\lambda = 0, k = 0, c \neq 0$; $\lambda = 0, k = \pm 1, c = 0$; $\lambda \neq 0, k = 0, c = 0$ (see Ref. [26] for the quantization of the corresponding minisuperspaces within the path integral formulation). It is easy to show that the coordinate change

$$x \equiv \frac{1}{2}(a\Omega + b\phi) \qquad y \equiv \frac{1}{2}(b\Omega + a\phi) \tag{26}$$

leads to the following form of the constraint:

$$H = -p_x^2 + p_y^2 + \zeta e^{2x} = 0, \tag{27}$$

with $sgn(\zeta) = sgn(A/(a^2 - b^2))$. The Wheeler-DeWitt equation corresponding to this constraint is

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \zeta e^{2x}\right)\Psi_\omega(x, y) = 0. \tag{28}$$

The general solution for the case $\zeta > 0$ is

$$\begin{aligned} \Psi_\omega(x, y) &= \left[a_+(\omega)e^{i\omega y} + a_-(\omega)e^{-i\omega y}\right] \\ &\quad \times \left[b_+(\omega)J_{i\omega}(\sqrt{|\zeta|}e^x) + b_-(\omega)N_{i\omega}(\sqrt{|\zeta|}e^x)\right], \end{aligned} \tag{29}$$

with $J_{i\omega}$ and $N_{i\omega}$ the Bessel and Neumann functions of imaginary order respectively. Instead, for $\zeta < 0$, the general solution is

$$\begin{aligned} \Psi_\omega(x, y) &= \left[a_+(\omega)e^{i\omega y} + a_-(\omega)e^{-i\omega y}\right] \\ &\quad \times \left[b_+(\omega)I_{i\omega}(\sqrt{|\zeta|}e^x) + b_-(\omega)K_{i\omega}(\sqrt{|\zeta|}e^x)\right], \end{aligned} \tag{30}$$

where $I_{i\omega}$ and $K_{i\omega}$ are modified Bessel functions. The obtention of these solutions to the Wheeler–DeWitt equation does not finish the quantization process given that we do not know yet which are the proper boundary conditions that we should impose on these spaces of solutions.

Instead of quantizing the model by solving the Wheeler–DeWitt equation, one could also reduce the system by identifying a time variable and solving the corresponding Schrödinger equations. Depending on the sign of the constant ζ in the constraint (27), these models admit as global phase time the coordinates x or y. In the case $\zeta > 0$ the

time is $t = \pm x$, so that we can define the reduced Hamiltonian as $h = \sqrt{p_y^2 + \zeta e^{2x}}$, being this reduced Hamiltonian time dependent. If, instead, we have $\zeta < 0$, the time is $t = \pm y$ and the reduced Hamiltonian is $h = \sqrt{p_x^2 - \zeta e^{2x}}$. In the first case, $\zeta > 0$ ($t = \pm x$), the corresponding Schrödinger equations are

$$i\frac{\partial}{\partial x}\Psi(x,y) = \pm\left(-\frac{\partial^2}{\partial y^2} + \zeta e^{2x}\right)^{1/2}\Psi(x,y) \tag{31}$$

In the second case ($\zeta < 0$, $t = \pm y$) the associated Schrödinger equations are

$$i\frac{\partial}{\partial y}\Psi(x,y) = \pm\left(-\frac{\partial^2}{\partial x^2} - \zeta e^{2x}\right)^{1/2}\Psi(x,y). \tag{32}$$

For both $\zeta > 0$ and $\zeta < 0$ we have a pair of Hilbert spaces, each one with its corresponding Schrödinger equation, and a conserved positive-definite inner product allowing for the usual probability interpretation of the wave function (this is analogous to the obtention of two quantum propagators, one for each disjoint theory, in the context of path integral quantization [47, 6]).

It is important to remark that in the present model the time variables which permit to reduce the system ($t = \pm x$ and $t = \pm y$ for the cases $\zeta > 0$ and $\zeta < 0$ respectively) can be selected among the original canonical coordinates, being then unnecessary to perform any kind of canonical transformation. In cases like this one the Schrödinger quantization formalism can be applied directly from the beginning in a straightforward manner.

It is also important to insist on the fact that the cases $\zeta > 0$ and $\zeta < 0$ are not symmetric: if in the case $\zeta < 0$ ($t = \pm y$) the reduced Hamiltonian $h = h(x, p_x)$ is time independent, in the case $\zeta > 0$ ($t = \pm x$) the reduced Hamiltonian $h = h(x, p_y)$ is time dependent.

For the case $\zeta < 0$, the stationary solutions to the Schrödinger equations corresponding to both sheets ($t = \pm y$) are

$$\Psi_\pm^\omega(x,y) = e^{\mp i\omega y}\left[b_+(\omega)J_{i\omega}(\sqrt{|\zeta|}e^x) + b_-(\omega)N_{i\omega}(\sqrt{|\zeta|}e^x)\right] \tag{33}$$

It is clear from this expression that the space of solutions (30) of the Wheeler–DeWitt equation for $\zeta < 0$ is the direct sum of the spaces of solutions of the Schrödinger equations

corresponding to each sheet of the Hamiltonian constraint. For the time dependent case ($\zeta > 0, t = \pm x$) the relation between both spaces of solutions is not so clear due to the presence of operator ordering ambiguities (this case will be discussed in Section 3.5).

Another point to be remarked is that as a result of the right definition of time, in both cases the reduced Hamiltonian h are real, so that the evolution operator is self-adjoint and the resulting quantization is unitary. Instead, a wrong choice of time, like for example $t = \pm x$ in the case $\zeta < 0$, leads to a reduced Hamiltonian h which is not real for all allowed values of the variables, and we obtain a non unitary theory. Another remarkable aspect is that a right time is not necessarily associated to the geometrical degrees of freedom, as one could naively expect. In the case $\lambda \neq 0, k = 0, c = 0$ for example, one obtains that the *geometrical* coordinate Ω is the physical clock and that the reduced Hamiltonian involves only the dilaton and the antisymmetric field (the last one, through the constant λ). On the contrary, in the case $\lambda = 0, k = 1, c = 0$ the physical clock is the dilaton, while the reduced Hamiltonian depends only on the geometry (see Section 4.1).

3.3 Extrinsic time

We shall now consider the problem of models not admitting a global time defined only in terms of the coordinates. A good example is given by a nontrivial dilaton cosmology described by the scaled Hamiltonian constraint

$$H = -p_\Omega^2 + p_\phi^2 + 2ce^{6\Omega+\phi} + \lambda^2 e^{-2\phi} = 0, \tag{34}$$

which corresponds to a flat universe with dilaton field ϕ and non vanishing antisymmetric field $B_{\mu\nu}$ coming from the NS-NS sector of effective string theory. This model is not solvable by separating variables, and in the case $c < 0$ the potential can vanish, so that it does not admit an intrinsic time. However, because these cosmologies come from the low energy string theory, which makes sense in the limit $\phi \to -\infty$, then the $e^\phi \equiv V(\phi)$ factor in the first term of the potential verifies $V(\phi) = V'(\phi) \ll 1$, and we can replace ce^ϕ by the constant $\overline{c}$ fulfilling $|\overline{c}| \ll |c|$:

$$H = -p_\Omega^2 + p_\phi^2 + 2\overline{c}e^{6\Omega} + \lambda^2 e^{-2\phi} = 0. \tag{35}$$

Following Ref. [37], we shall start in a naive way by quantizing by means of a Wheeler–DeWitt equation the model described by the Hamiltonian constraint (35). However, for this model we can also perform a canonical transformation which permits to obtain a Hamiltonian constraint that can be factorized in a two-sheet constraint of the form $H = (p_0 + h)\,(p_0 - h) = 0$ [12]. We will then be able to quantize the model also by using the Schrödinger equations corresponding to each sheet of the factorized Hamiltonian constraint. It is important to remark that this Schrödinger quantization is not trivial in the sense that a non trivial canonical transformation is needed. If this canonical transformation were not known, we should face the problem of imposing boundary conditions on the space of solutions of the Wheeler–DeWitt equation. Given that this model can be quantized by both methods, it is of great value in order to understand how to impose boundary conditions to the solutions of the Wheeler–DeWitt equation for cases in which a global time is not known.

The Wheeler–DeWitt equation associated to the constraint (35) is

$$\left(\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\phi^2} + 2\overline{c}e^{6\Omega} + \lambda^2 e^{-2\phi}\right)\Psi(\Omega,\phi) = 0, \tag{36}$$

and its solutions are

$$\begin{aligned}\Psi_\omega(\Omega,\phi) \;&=\; \left[a(\omega)I_{i\omega}\left(|\lambda|e^{-\phi}\right) + b(\omega)K_{i\omega}\left(|\lambda|e^{-\phi}\right)\right] \\ &\quad\times\left[c(\omega)I_{i\omega/2}\left(\sqrt{|2\overline{c}|}e^{3\Omega}\right) + d(\omega)K_{i\omega/2}\left(\sqrt{|2\overline{c}|}e^{3\Omega}\right)\right],\end{aligned} \tag{37}$$

where I_ν and K_ν are modified Bessel functions. In principle one does not know which are the proper boundary conditions that should be imposed on these solutions. If one was to proceed in a naive way without any consideration about time, then both contributions including the functions I_ν should be discarded, because they diverge in what would be commonly understood as a region classically forbidden by the behavior of the exponential terms in the potential. In fact, this has been the choice in the case of the Taub universe in Ref. [37]. However, as we shall show, the functions $I_{i\omega}(|\lambda|e^{-\phi})$ should not be discarded, because in the picture including a globally right notion of time, the dilaton ϕ is associated to the physical clock.

This can be clearly realized by performing the canonical transformation introduced for the Taub universe in Ref. [12] in order to obtain a constraint with only one term in the potential. This is achieved by introducing the generating function of the first kind

$$\Phi_1(\phi, s) = \pm|\lambda|e^{-\phi}\sinh s. \tag{38}$$

The new canonical variables are then given by

$$\begin{aligned} s &= \pm\text{arcsinh}\left(\frac{p_\phi e^{\phi}}{|\lambda|}\right) \\ p_s &= \pm|\lambda|e^{-\phi}\cosh s. \end{aligned} \tag{39}$$

With this canonical transformation the resulting form for the Hamiltonian constraint in the limit $V(\phi) = V'(\phi) \ll 1$ is

$$H = -p_\Omega^2 + p_s^2 + 2\overline{c}e^{6\Omega} = 0, \tag{40}$$

and we can apply our procedure starting from this constraint. In the case $\overline{c} < 0$, the momentum p_s does not vanish and the time is $t = \pm s$. According to the definition of the new variable s the time $t = \pm s$ is a function of both p_ϕ and ϕ, being then an *extrinsic time*. Once again, observe that, differing from what is sometimes considered the "natural" choice, the physical clock is not associated to the metric, but to the matter field (see below). The constraint (40) can be written as

$$H = \left(p_s - \sqrt{p_\Omega^2 - 2\overline{c}e^{6\Omega}}\right)\left(p_s + \sqrt{p_\Omega^2 - 2\overline{c}e^{6\Omega}}\right) = 0 \tag{41}$$

(see Section 3.5). If this constraint equation is satisfied by demanding $p_s = h = \sqrt{p_\Omega^2 + |2\overline{c}|e^{6\Omega}}$, then this choice implies (taking into account that $p_t = -h$) that one has selected the variable $-s$ to play the role of the physical clock ($t = -s$ and $p_t = -p_s$). If on the contrary one demands $p_s = -h = -\sqrt{p_\Omega^2 + |2\overline{c}|e^{6\Omega}}$, then one is choosing s as the physical clock ($t = s$ and $p_t = p_s$). In both cases the positive definite reduced Hamiltonian h does not depend on the time variable $t = \pm s$. As we will explain later, the fact that the reduced Hamiltonian h is time independent assures the equivalence at the quantum level

of the factorized constraint (41) and the original one given by (40). The corresponding Schrödinger equations associated to the choices $t = s$ and $t = -s$ are respectively

$$i\frac{\partial}{\partial s}\Psi_+(\Omega, s) = \left(-\frac{\partial^2}{\partial\Omega^2} + |2\overline{c}|e^{6\Omega}\right)^{1/2} \Psi_+(\Omega, s). \tag{42}$$

$$-i\frac{\partial}{\partial s}\Psi_-(\Omega, s) = \left(-\frac{\partial^2}{\partial\Omega^2} + |2\overline{c}|e^{6\Omega}\right)^{1/2} \Psi_-(\Omega, s). \tag{43}$$

By $\Psi_\pm$ we have noted the solutions corresponding to the sheets $K_\pm = p_s \pm h = 0$. Note that $\Psi_+ = (\Psi_-)^*$ (this relation between both spaces of solutions will be discussed later). Given that the reduced Hamiltonian $h = \sqrt{p_\Omega^2 - 2\overline{c}e^{6\Omega}}$ does not depend on time, one can propose a solution of the general form $\Psi_E(\Omega, t) = \varphi_E(\Omega)\, e^{-iEt}$, where $\varphi_E(\Omega)$ satisfies the time independent Schrödinger equation $\hat{h}(\Omega)\,\varphi_E(\Omega) = E\varphi_E(\Omega)$. Given that $\hat{h}$ is a square root operator, one has to use the spectral theorem and solve the derived equation $\hat{h}^2(\Omega)\,\varphi_E(\Omega) = E^2\varphi_E(\Omega)$ [8]. The general solution to this last equation is

$$\varphi_\omega(\Omega) = \left[c(\omega)I_{i\omega/2}\left(\sqrt{|2\overline{c}|}e^{3\Omega}\right) + d(\omega)K_{i\omega/2}\left(\sqrt{|2\overline{c}|}e^{3\Omega}\right)\right] \tag{44}$$

The functions $I_{i\omega/2}\left(\sqrt{|2\overline{c}|}e^{3\Omega}\right)$ have to be discarded because they diverge in the classical forbidden zone $\Omega \to \infty$. The stationary solutions of the Schrödinger equations (42) and (43) are then of the form

$$\Psi_\pm^\omega(\Omega, s) = d(\omega)K_{i\omega/2}\left(\sqrt{|2\overline{c}|}e^{3\Omega}\right)e^{\mp i\omega s} \tag{45}$$

It is important to remark that the equation $\hat{h}^2(\Omega)\,\varphi_E(\Omega) = E^2\varphi_E(\Omega)$ is the same that one obtains from the Wheeler-DeWitt equation associated to the constraint (40) by separating variables (i.e., by proposing solutions of the form $\Psi(\Omega, s) = \varphi(\Omega)\,\phi(s)$). This means that the equation that one has to solve *in any case* is a second order Wheeler-DeWitt equation of the hyperbolic type. The reduction process changes only the variables in which this equation will be solved. The only difference is that in the new set of variables it is clear which are the boundary conditions to be imposed.

3.4 Relation between solutions of the Schrödinger and the Wheeler-DeWitt equations

As we have shown in the preceding examples, there are mainly two quantization schemes. Being the Wheeler-DeWitt equation an hyperbolic equation quadratic in all its derivatives, its space of solutions is twice the space of solutions of the parabolic Schrödinger equation linear in $\frac{\partial}{\partial t}$. If the reduced Hamiltonian h does not depend on time, the space of solutions of the Wheeler-DeWitt equation $\widehat{H}\Psi = 0$ will be simply the direct sum of the spaces of solutions of the Schrödinger equations corresponding to each sheet of the Hamiltonian constraint:

$$Ker\widehat{H} = Ker\widehat{K_+} \oplus Ker\widehat{K_-}, \tag{46}$$

where $\widehat{K_+}$ and $\widehat{K_-}$ are the operators corresponding to the factors $p_0 + h$ and $p_0 - h$ respectively.

It could happen nevertheless that, even if the reduced Hamiltonian h is time independent, the correspondence between the solutions of the Wheeler-DeWitt equation and the solutions of the Schrödinger equation is not so direct. For the dilaton cosmology described by the Hamiltonian constraint (34), the passage from the Hamiltonian constraint in its original form to the factorized form (41) is mediated by the canonical transformation given by (39). The solutions (37) of the Wheeler-DeWitt equation (36) and the solutions of the Schrödinger equations (42) and (43) are not expressed in terms of the same variables $((\Omega, \phi)$ and (Ω, s) respectively). In Ref. [12], this situation was analyzed for the Taub model and a certain criterium was proposed for fixing boundary conditions to the Wheeler-DeWitt equation for those cases where we do not know how to reduce the system. We will now describe such a proposal.

In Ref. [12] it was proposed that the solutions of the Wheeler-DeWitt equation (36) can be related with the solutions of the Schrödinger equations (42) and (43) by means of a quantum version of the classical canonical transformation (39) used to reduce the system. Under certain conditions two quantum systems whose Hamiltonians are canonically equivalent at the classical level, have as quantum states wave functions which can be

related by means of the so called “quantum canonical transformations”. This quantum version of the classical canonical transformations can be understood as a generalization of the Fourier Transform. It is in fact possible to consider the Fourier transform as the quantum version of the classical canonical transformation which interchanges coordinates and momenta. The generating function of such a canonical transformation is $F_1(q,Q) = Qq$ and the equations defining the transformation are

$$p = \frac{\partial F_1}{\partial q} = Q, \qquad P = -\frac{\partial F_1}{\partial Q} = -q.$$

The Fourier transform

$$\Psi(q) = N \int dp e^{ipq} \Phi(p) \tag{47}$$

can then be rewritten as

$$\Psi(q) = N \int dQ e^{iF_1(q,Q)} \Phi(Q), \tag{48}$$

The inverse of this transformation could be expressed as

$$\Phi(Q) = N \int dq \left| \frac{\partial^2 F_1(q,Q)}{\partial q \partial Q} \right| e^{-iF_1(q,Q)} \Psi(q). \tag{49}$$

It is then natural to ask if these expressions remain valid for a canonical transformation given by a general generating function $F_1(q,Q)$. If this were the case, one would have a Generalized Fourier Transform between the quantum representations associated to systems canonically equivalent at the classical level. In general this is not the case: certain conditions must be fulfilled in order to have this kind of Generalized Fourier Transforms. In [25] it was shown that the expression (48) is valid if the following condition is satisfied

$$H_q\left(-i\frac{\partial}{\partial q}, q\right) e^{iF_1(q,Q)} = H_Q\left(i\frac{\partial}{\partial Q}, Q\right) e^{iF_1(q,Q)}, \tag{50}$$

where certain boundary conditions in the integration limits are also assumed. If the canonical transformation is defined by means of other kind of generating function (F_2, F_3 or F_4) analogous expressions can be used.

In particular, the canonical transformation (39) used to pass from the Hamiltonian constraint (34) to the factorized Hamiltonian constraint (41) effectively satisfies these

requirements. The solutions of the Schrödinger equations (42) and (43) can then be related to the solutions of the Wheeler-DeWitt equation (36) by means of the corresponding Generalized Fourier Transform.

By means of this formalism we can now analyze the proper boundary conditions that should be imposed on the space of solutions of the Wheeler-DeWitt equation (36). Firstly, one should notice that the dependance on Ω is the same for both representations ((Ω, ϕ) and (Ω, s)). This means that one can apply, on the factor depending on Ω in the solutions (37) of the Wheeler-DeWitt equation, the same boundary conditions that we have previously imposed on the solutions of the Schrödinger equations. In fact, given our choice of the physical clock as a function of both ϕ and p_ϕ (see (39)), the variable Ω is an authentical dynamical variable. In this way one can discard in the solution (37) the functions $I_{i\omega/2}\left(\sqrt{|2\overline{c}|}e^{3\Omega}\right)$ because they diverge in the classical forbidden zone $\Omega \to \infty$. Next we have to impose boundary conditions on the factor which depends on ϕ in (37), i.e., we have to decide if we will discard the functions $I_{i\omega}\left(|\lambda|e^{-\phi}\right)$ or the functions $K_{i\omega}\left(|\lambda|e^{-\phi}\right)$. The criterium that we will apply is that the physical solutions will be those whose transformed functions (via the generalized Fourier transform) coincide with the factors $e^{-i\omega s}$ or $e^{i\omega s}$ in the solutions $\Psi\pm$ (45) of the Schrödinger equations (42) and (43) for the clock choices $t = s$ or $t = -s$ respectively. One could suppose that the correct solutions will be those which go to zero in the classical forbidden zone $\phi \to -\infty$, i.e., the functions $K_{i\omega}\left(|\lambda|e^{-\phi}\right)$. But if we transform this functions we have to conclude that they have to be discarded given that they correspond to a linear combination of the form $ae^{i\omega s}+be^{-i\omega s}$. If on the contrary we apply the generalized Fourier transform to the functions $I_{\pm i\omega}\left(|\lambda|e^{-\phi}\right)$ we obtain the correspondence $I_{\pm i\omega}\left(|\lambda|e^{-\phi}\right) \leftrightarrow e^{\mp i\omega s}$. The functions $I_{i\omega}\left(|\lambda|e^{-\phi}\right)$ and $I_{-i\omega}\left(|\lambda|e^{-\phi}\right)$ do represent then the positive energy states corresponding to the clock choices $t = s$ and $t = -s$ respectively. If we make the choice $t = s$ the solution to the Wheeler-DeWitt equation (36) is then

$$\Psi_\omega(\Omega, \phi) = \tilde{a}\left(\omega\right) I_{i\omega}\left(|\lambda|e^{-\phi}\right) K_{i\omega/2}\left(\sqrt{|2\overline{c}|}e^{3\Omega}\right) \tag{51}$$

The fact that the correct solutions are not those which go to zero in the classical

forbidden zone $\phi \rightarrow -\infty$ does not pose a problem given that the variable ϕ is not a dynamical variable, but the variable associated with the physical clock that we have chosen. This fact constitutes an important difference with the immediate result that we would obtain by following Ref. [37].

It could be argued that the real problem is to find proper boundary conditions for the Wheeler-DeWitt equation when one does not know how to reduce the system. If one has a reduced system with the corresponding Schrödinger equation as in the Taub case, it is no more necessary to go back to the Wheeler-DeWitt "representation". However, given that for the Taub model both spaces of solutions are known, its analysis is of great utility in order to propose general boundary conditions to the Wheeler-DeWitt equation, even for those cases for which one does not know how to separate a physical clock. The main requirement to impose on the general boundary conditions that we are looking for is that, for those cases for which one knows how to reduce the model (as the Taub universe), they have to select the same quantum states imposed by the quantization of the reduced system. In Ref. [12] certain steps were given in this direction. We will now describe these results.

Let us suppose as a first simplification that we have the following scaled Hamiltonian constraint

$$H = p_0^2 + V\left(q^0\right) - h^2\left(q^\mu, p_\mu\right) \tag{52}$$

with $V\left(q^0\right) > 0$. In other words, let us suppose that there is not a non-minimal coupling between a certain variable q^0 and the rest of the canonical variables q^μ. Given this form of the Hamiltonian constraint, the solutions of the corresponding Wheeler-DeWitt equation will have the form $\Psi\left(q^0, q^\mu\right) = \Theta\left(q^0\right)\Phi\left(q^\mu\right)$. If the potential $V\left(q^0\right)$ were identically zero, the variable q^0 would be a physical clock. We could suppose that the true physical clock will be a certain function of q^0 which, in the region where $V\left(q^0\right) \rightarrow 0$, coincides with q^0. We can expect then that the solutions $\Psi\left(q^0, q^\mu\right)$ of the Wheeler-DeWitt equation corresponding to (52) would tend, in the region where $V\left(q^0\right) \rightarrow 0$, to wave functions of the form $e^{-iEq^0}\Phi\left(q^\mu\right)$. The boundary conditions to be imposed on the space of solutions of

the Wheeler-DeWitt equation is that the physical solutions will be those functions whose asymptotic expressions in the region where $V(q^0) \to 0$ are of the form $e^{-iEq^0}\Phi(q^\mu)$ (with $\Phi(q^\mu)$ going to zero in the classical forbidden zone associated to q^μ). If we do not know which is the physical clock for reducing the system, but we know that there is a variable q^0 which is a physical clock *in a certain limited region* with a time independent reduced Hamiltonian h, then we can imposed as boundary conditions that the physical solutions have to tend in that region to functions with a factor e^{-iEq^0}.

If we now apply this criterium to the solutions of the Wheeler-DeWitt equation (36) corresponding to the Taub model, we will select the correct quantum states, i.e. the quantum states (51) selected by the reduction of the model. The Hamiltonian constraint (35) of the Taub model has in fact the form (52). In the region where the potential term $\lambda^2 e^{-2\phi}$ goes to zero, i.e., in the region $\phi \to \infty$, the variable ϕ is a physical clock. Following the proposed criterium, we have to select those functions which tend to functions with a factor $e^{-iE\phi}$ in the region $\phi \to \infty$. If we consider the asymptotic expressions of the functions $K_{i\omega}\left(|\lambda| e^{-\phi}\right)$ and $I_{\pm i\omega}\left(|\lambda| e^{-\phi}\right)$ in that limit, we will find that the former tend to a linear combination of the form $ae^{iE\phi} + be^{-iE\phi}$, while only the latter tend to functions of the form $e^{\pm iE\phi}$.

3.5 Time dependent reduced Hamiltonians

A serious problem for the understanding of these two quantization formalisms and its relations appears when the reduced Hamiltonian h is time dependent. While the constraints $H = p_0^2 - h^2 = 0$ and $H = (p_0 + h)(p_0 - h) = 0$ are classically equivalent, at the quantum level this equivalence is no more fulfilled if the reduced Hamiltonian h depends on the variable chosen as physical clock. In fact a wave function in the kernel of the operators $\widehat{K}_+$ and $\widehat{K}_-$ corresponding to the factors $(p_0 + h)$ and $(p_0 - h)$ respectively (i.e., a solution of the Schrödinger equation), is not necessarily annihilated by the operator $\widehat{H}$ associated with the Wheeler-DeWitt equation. If the reduced Hamiltonian h is time dependent the product of the Schrödinger operators $\widehat{K}_+$ and $\widehat{K}_-$ is not equal to the Wheeler-DeWitt

operator $\widehat{H}$. The two possible products of the Schrödinger operators are

$$\widehat{K}_{\pm}\widehat{K}_{\mp} = -\frac{\partial^2}{\partial t^2} - \hat{h}^2 \mp \left[-i\frac{\partial}{\partial t}, \hat{h}\right] \tag{53}$$

If $h \neq h(t)$ (i.e., $[p_t, h] = 0$), then $\widehat{K}_{+}\widehat{K}_{-} = \widehat{K}_{-}\widehat{K}_{+} = \widehat{H} = -\frac{\partial^2}{\partial t^2} - \hat{h}^2$. If $h = h(t)$, then the Wheeler-DeWitt operator $\hat{H}$ is equal to

$$\widehat{H} = \frac{1}{2}\left(\widehat{K}_{+}\widehat{K}_{-} + \widehat{K}_{-}\widehat{K}_{+}\right). \tag{54}$$

It is then clear that the solutions to the Schrödinger equations $\widehat{K}_{\pm}\Psi_{\pm} = 0$ are not necessarily solutions of the Wheeler-DeWitt equation $\widehat{H}\Psi = 0$ when the reduced Hamiltonian h is time dependent [8]. As it was explained in [8], if the reduced Hamiltonian h is time dependent the solution to the Schrödinger equation $i\frac{\partial \Psi(x,t)}{\partial t} = \hat{h}\left(x, p, t\right)\Psi\left(x, t\right)$ takes the form

$$\Psi\left(x, t\right) = T\left[e^{-i\int_{t_0}^{t}\hat{h}(t')dt'}\right]\Psi\left(x, t_0\right), \tag{55}$$

where T is the time-ordering operator. If the condition

$$\left[h\left(t\right), h\left(t'\right)\right] = 0 \tag{56}$$

is satisfied, the expression (55) gives

$$\Psi\left(x, t\right) = e^{-i\int_{t_0}^{t}\hat{h}(t')dt'}\Psi\left(x, t_0\right). \tag{57}$$

The condition (56) implies also that there exists a conserved complete set of eigenstates of the Hamiltonian operator $\hat{h}\left(t\right)$, i.e., a set of basis states which are eigenstates of $\hat{h}$ at all times. If $\Psi_E\left(x\right)$ is an eigenstate of $\hat{h}\left(t_0\right)$, then $\Psi_E\left(x\right)$ will be an eigenstate of $\hat{h}\left(t\right)$ for all times, i.e., there will be a function $E\left(t\right)$ such that

$$\hat{h}\left(t\right)\Psi_E\left(x\right) = E\left(t\right)\Psi_E\left(x\right). \tag{58}$$

The time evolution of such a state is given by

$$\Psi_E\left(x, t\right) = e^{-i\int_{t_0}^{t}E(t')dt'}\Psi_E\left(x, t_0\right). \tag{59}$$

An example of a system with a time dependent reduced Hamiltonian $h(t)$ is provided by the dilatonic cosmological model corresponding to the Hamiltonian constraint (27) for $\zeta > 0$. The factorized form of this constraint is

$$H = \left(-p_x + \sqrt{p_y^2 + \zeta e^{2x}}\right)\left(p_x + \sqrt{p_y^2 + \zeta e^{2x}}\right) = 0, \tag{60}$$

being the potential time dependent for $t = \pm x$. Therefore, though at the classical level this product is equivalent to the constraint (27), in its quantum version both constraints differ in terms associated to commutators between $\hat{p}_x$ and the potential ζe^{2x}. The general form of these commutators is $\left[\sqrt{\sum(\hat{p_\mu})^2 + V(\hat{q^i})}, \hat{p}_0\right]$ (where $\mu \neq 0$, and V stands for the potential in the scaled Hamiltonian constraint H). Hence, depending on which of the two classically equivalent constraints we start from, we obtain different quantum theories. Observe that this problem appears in the case for which the Wheeler-DeWitt equation leads to a result in which the usual identification of positive and negative-energy solutions is not apparent, at least in the standard form of exponentials of the form $e^{i\omega t}$. In this sense, recall the difference between the solutions (29) and (30).

The central obstruction for the existence of a trivial correspondence between the Wheeler-DeWitt and Schrödinger solutions for minisuperspaces is then a constraint with a time-dependent potential. For the class of models of Section 3.2, a coordinate choice avoiding the decision between inequivalent quantum theories can be introduced [47]. Consider the constraint (25) and define

$$u = \alpha e^{\frac{1}{2}(aq^1 + bq^2)} \cosh\left(\frac{bq^1 + aq^2}{2}\right), \qquad v = \alpha e^{\frac{1}{2}(aq^1 + bq^2)} \sinh\left(\frac{bq^1 + aq^2}{2}\right), \tag{61}$$

with $\alpha = \sqrt{|A|}$. These coordinates allow to write the scaled constraint in the equivalent form

$$H = -p_u^2 + p_v^2 + \eta m^2 = 0, \tag{62}$$

with $\eta = sgn(A)$ and $m^2 = 4/|a^2 - b^2|$. Given that the commutators do not appear now, the Wheeler-DeWitt equation is equivalent to the corresponding Schrödinger equations, being the physical clock the coordinates u or v depending on η: for $\eta = 1$ we have $p_u \neq 0$ and $t = \pm u$, while for $\eta = -1$ we have $p_v \neq 0$ and $t = \pm v$.

Of course, such a solution can be applied only for a limited number of minisuperspace models. We believe that the case of a time dependent reduced Hamiltonian $h = h(t)$ is the general case. In fact it seems highly improbable that one could find a degree of freedom for playing the role of a physical clock which would not be coupled to the others degrees of freedom (besides the minimal coupling given by the Hamiltonian constraint). In other words, we would not expect to find a "free clock", i.e. a degree of freedom without a non-minimal coupling with the others degrees of freedom. The reduced system will be then in general an open system, i.e., a system which interacts with the clock [17]. If the reduced Hamiltonian h is time dependent, the energy of the reduced system will change. This fact could seem bizarre if one takes into account that in quantum cosmology we are considering the whole universe. But in that case the physical clock will have its own energy, being the change in the energy of the reduced system a consequence of its interaction with the clock (which is nothing but a particular interacting degree of freedom). This change in the "reduced" energy of the system will correspond then to a variation in the momentum of the physical clock ($p_t = -h$): if the clock changes its rate of evolution, the total energy of the others degrees of freedom will change.

4 General aspects of the problem of time

4.1 Parametrized system formalism

A general framework for understanding the meaning of the Hamiltonian constraint $H = 0$ is provided by the formalism of parametrized systems (for details, see [31]; for recent developments, see for example [21, 48, 12, 11, 2, 51, 3]). In this formalism the "real" time t is added to the canonical variables of an ordinary dynamical system, being the increased set of variables left as a function of a physically irrelevant parameter τ. The ordinary action of a dynamical system

$$S\left[q^{\mu}, p_{\mu}\right] = \int_{t_1}^{t_2} p_{\mu} dq^{\mu} - h\left(q^{\mu}, p_{\mu}\right) dt, \quad \mu = 1, ..., n \tag{63}$$

can then be converted into a parametrized action by means of the definition of a new pair of canonical variables $\{q^0 = t, p_0 = -h\}$ associated to the time t and the Hamiltonian h.

The extended set of variables are left as functions of the physically irrelevant parameter τ. The set $\{q^0, q^\mu, p_0, p_\mu\}$ can be varied independently, provided that the definition of p_0 is incorporated into the action as a constraint

$$H = p_0 + h = 0, \tag{64}$$

with the corresponding Lagrange multiplier N (lapse function). In this way one obtains the following action

$$S\left[q^i(\tau), p_i(\tau), N(\tau)\right] = \int_{t_1}^{t_2} \left(p_i \frac{dq^i}{d\tau} - NH\right) d\tau, \quad i = 0, ..., n \tag{65}$$

The presence of the Lagrange multiplier $N(\tau)$ means that the dynamics remains ambiguous in the irrelevant parameter τ (one could say that it has no sense to speak about dynamics until the hidden time is recovered). In the extended configuration space $\{q^0 = t, q^\mu\}$ the solutions of the equations of motions are static "curves" without a preferred parametrization. The theory is then invariant under reparametrizations of the physical irrelevant parameter τ. We could say that it has no sense to speak about the speed of the motion of the system through its extended configuration space. These curves have nevertheless a preferred sense: they unfold in the increasing directions of $q^0 = t$.

In this way any dynamical system can be formulated in the framework of parametrized systems. In the process of parametrization, the real time is "disguised" as a dynamical variable. This disguised time can be nevertheless easily recognized due to the linearity of its conjugated momentum in the constraint (64).

It has been intended to use the formalism of parametrized system as a model for understanding the canonical structure of General Relativity. Having the theory a Hamiltonian constraint of the form $H = 0$, one could suppose that General Relativity is an ordinary dynamical theory whose action is presented in a parametrized form. If that were the case, one could try to reduce the system by finding the disguised "real" time and formulating the theory as an ordinary dynamical system of the form (63), whose evolution takes place in that "real" time variable. There has been many proposals for that hidden time variable but none of them could circumvent the different problems which appear in the reduction

process [33, 34]. Besides the lack of a universal "real" time variable, there is still another important objection against this interpretation of the canonical structure of General Relativity: the constraint (64) is linear in the momentum p_0, while the Hamiltonian constraint of General Relativity is quadratic in all its momenta.

In [11] a different interpretation was proposed in order to circumvent this problem. Here we will follow this new conceptual framework. We will consider that the supposition of a privileged "real" time hidden among the canonical variables is an unfounded supposition. We consider that one of the most fundamental properties of General Relativity is that its solutions do not represent an *evolution in time* of certain dynamical variables, but that it is a theory which selects certain relative (not dynamical) configurations of its canonical variables which, under certain conditions, can be considered as dynamical evolutions if proper *physical clocks* are selected. In fact we can never observe the evolution of the degrees of freedom along a "newtonian" time flow like $q^1(t)$ and $q^2(t)$ but rather the evolution of certain variables relative to the change of another variable, i.e., something like $q^2(q^1)$. Following this interpretation, there would not be such a thing as Time, but only physical degrees of freedom which, under certain conditions, can be used as physical clocks. In this relational approach we cannot say that reducing the system means to find the hidden "real" time: in order to reduce the model we have only to select a monotonously increasing canonical variable as a physical clock. It is thus only possible to speak about physical clocks, i.e., degrees of freedom which, under certain conditions, can play the role of evolution parameters for the others degrees of freedom. In that case it is not important if the physical clock is a geometrical degree of freedom or a degree of freedom associated to the matter fields (see, for example, [9]); in fact, among the examples of sections 3.2 and 3.3, we identified models with the dilaton field playing the role of physical clock. On the contrary, if we were looking for the "real" time, we could expect that this hidden time would be a geometrical degree of freedom.

The works [30, 21, 44, 45, 18] have shown that the general covariance of General Relativity is not so different from the gauge symmetry of an ordinary gauge theory (see Appendix B). In fact, we could consider General Relativity as a particular case of a gauge

theory. In general, in gauge theories the gauge fixing is not interpreted as the discovery of the “real” degrees of freedom, but as a particular choice without any “god given” privilege. In the framework of gravitation the election of a physical clock acts in fact as a partial gauge fixing of the theory (election of the Lagrange multiplier $N(\tau)$). If we follow the ordinary interpretation of gauge theories this election should not be interpreted as the discovery of the real hidden time, but as a choice of a particular reference system for measuring the evolution of the system.

We could now ask which is the proper formalism for implementing such an interpretation. In [11] a certain modification of the parametrized formalism was proposed in order to explore the consequences of this new conceptual framework.

The first point that one should take into account is that, as there is not a privileged time variable t and a privileged conjugated momentum $p_t = -h$, all the momenta must appear on an equal footing in the Hamiltonian constraint (as effectively happens in General Relativity). This means that there should not be a preferred momentum appearing linearly as in (64). It is then a consequence of the new interpretation the necessity of finding a formalism where all the momenta appear quadratically in the Hamiltonian constraint.

There is still a second argument for implementing a formalism of parametrized systems with a constraint quadratic in all its momenta. The solutions of the theory are static trajectories in a certain configuration space. In order to describe these static trajectories as dynamical evolutions, we have to choose a certain variable to play the role of time. The main difference with the original formalism of parametrized systems is that these trajectories do not carry a preferred sense of evolution. We can choose to unfold them in both directions. This ambiguity can be cast into the parametrized system formalism by means of a modified Hamiltonian constraint. If there is not a privileged time the solutions are necessarily static trajectories, i.e., relative configurations among the different variables, for example $q^1(q^0)$. If one wants now to select a physical clock, for example q^0 (we are supposing that q^0 is a monotonously increasing function along the trajectory) there is still an ambiguity: one still has to choose in which direction the trajectory will be unfold. This means that one can choose $t = q^0$ or $t = -q^0$. The static trajectory does not privilege any

direction and then both kinds of solutions must appear in the reduced formalism. These two options correspond to the constraints $K_+ = p_0 + h = 0$ and $K_- = p_0 - h = 0$. Both possibilities can be incorporated into the formalism if we use a Hamiltonian constraint of the form $H = (p_0 + h)(p_0 - h) = p_0^2 - h^2 = 0$. In order to make this factorization the original Hamiltonian constraint must be quadratic in the momentum conjugated to q^0. The existence of two sheets in the Hamiltonian constraint, far from being an unnecessary redundance, acquires in this way a precise meaning related to the necessity of having a parametrized formalism which does not privilege any sense of evolution. Each sheet of the Hamiltonian constraint $H = 0$ is then associated to each possible choice of the direction in which the trajectory can be unfold. The resulting action is

$$S\left[q^i(\tau), p_i(\tau), N(\tau)\right] = \int p_\mu dq^\mu + p_0 dq^0 - N\left(p_0^2 - h^2\right) d\tau \tag{66}$$

The constraint $H = 0$ is fulfilled if one of the factors vanishes on the constraint surface. To choose which factor is null is equivalent to choose which direction of q^0 is the increasing direction of time. This system can then be reduced by the clock choices $t = q^0$ or $t = -q^0$, corresponding these choices to the sheets $p_0 + h = 0$ and $p_0 - h = 0$ respectively. If $t = q^0$, then $p_t = p_0 = -h$; if $t = -q^0$, then $p_t = -p_0 = -h$. In both cases $p_t = -h$ with $h > 0$. Having fixed a factor to zero in the Hamiltonian constraint the other one will have, on the constraint surface, a definite sign, so being possible to rescale the Hamiltonian by this factor. If for example we fix $K_+ = p_0 + h = 0$ ($t = q^0$ as physical clock), we will have $p_0 = -h < 0$ and then the other factor will be $K_- = p_0 - h = -2h < 0$.

In this way, both the necessity of having a Hamiltonian constraint where all the momenta appear on an equal footing (quadratically) and the necessity of having the possibility of choosing the direction of unfolding of any static trajectory in both directions, conduce naturally to a modification of the parametrized formalism. We believe that this *quadratic parametrized formalism* is the proper formalism for understanding the canonical structure of General Relativity.

4.2 Motion-reversal and clock-reversal transformations

Without being precise, one could say each choice in the sense of evolution of the physical clock ($t = q^0$ or $t = -q^0$) corresponds to a kind of "time reversal" of the other one. But by choosing a physical clock, i.e., by choosing a sheet of the Hamiltonian constraint $H = (p_0 + h)(p_0 - h) = 0$, one obtains an ordinary dynamical system. It is well known that an ordinary classical or quantum system possess a certain symmetry under "time reversals". If each sheet posses a symmetry under "time reversals", the passage from one sheet to the other one can not be identified with the same time reversal operation. In [11] this situation was clarified. The result of this analysis is that there are two kind of "time reversals" operations which have to be carefully differentiated.

The first one is the ordinary "time reversal" operation of classical and quantum mechanics. In fact this operation does not correspond to an inversion in the direction of time, but to an inverted movement which unfolds in the same direction of time than the original solution. In [11] this operation was called *motion-reversal transformation*. Given a classical trajectory $\{q(q_0, p_0, t_0, t), p(q_0, p_0, t_0, t)\}$ which unfolds from $\{q_0, p_0\}$ at time t_0 to $\{q_f, p_f\}$ at time t_f, there exists a related trajectory which is also a solution of the same Hamilton equations. This inverted trajectory is

$$\begin{aligned} q^{mr}(q_0^{mr} = q_f, p_0^{mr} = -p_f, t_0, t) &= q(q_f, -p_f, t_0, t), \\ p^{mr}(q_0^{mr} = q_f, p_0^{mr} = -p_f, t_0, t) &= p(q_f, -p_f, t_0, t), \end{aligned} \tag{67}$$

and exists provided that the Hamiltonian h is quadratic in p and does not depend on t. This motion-reversed solution $\{q^{mr}, p^{mr}\}$ unfolds from $\{q_f, p_f\}$ at time t_0 to $\{q_0, p_0\}$ at time t_f. It is important to emphasize that this motion-reversed solution is a solution of the *same* Hamilton equations. This solution starts at the same time t_0 than the original one and unfolds in the same direction of time, but with initial conditions which have been inverted with respect to the original trajectory: the new trajectory starts with an inverted momentum from the point where the original one ends. This operation does not correspond to the operation of "playing the film backwards". In order to see this it is necessary to realize that, being the clock a dynamical variable, it has to be included

in the hypothetical film. If we play the film backwards, we will see the clock running backwards. However, we have just said that the motion-reversal transformation of a given solution unfolds in the same direction of time, i.e., the clock must continue to run forward. In other words, the motion-reversed solution corresponds, not to play the same film backwards but to play *another* film in which all behaves as running backwards, but the clock is still running forward. The motion-reversal transformation reverses then all variables, *but the variable identified with physical clock*. In the extended configuration space which includes the physical clock the result of a motion-reversal transformation it is not the same curve (i.e., the same film) unfolded in the opposite direction, but *another* curve which unfolds in the same direction of time.

On the other hand we have a symmetry associated to the passage from one sheet of the Hamiltonian constraint $H = (p_0 + h)(p_0 - h) = 0$ to the other one. This operation involves *also* a change in the direction of unfolding of the variable used as physical clock. This operation was called in [11] *clock-reversal transformation*. The clock-reversed solution will be a solution of the Hamilton equations corresponding to the other sheet of the Hamiltonian constraint. This operation does correspond now to the operation of "playing the film backwards". In the extended configuration space which includes the physical clock, the graph of the solution obtained by means of this operation coincides with the original one, being the only difference the sense of evolution. In other words, it is now the same film (i.e., the same curve in the extended configuration space) played in the reversed sense.

Given a solution of the equations of motion in one of the sheets of the Hamiltonian constraint, it is then possible to construct three others related solutions: the motion-reversal (in the same sheet as the original one), the clock-reversal (in the other sheet), and the motion reversal of the clock-reversed solution (in the other sheet). Summarizing, we have two solutions in each sheet.

The main difference between the linear formalism for parametrized systems and the quadratic one is that the latter is invariant under clock-reversal transformations. In general, any dynamical trajectory (which carries then a *fixed* sense of evolution) can be

considered as a static one by considering the clock in an extended configuration space. Inversely, any static trajectory (without a *fixed* sense of evolution) can be "temporalized" by considering a monotonously increasing variable along the trajectory as a physical clock. We could say that if the linear formalism is the proper formalism for considering the solutions of a dynamical system as static trajectories in an extended configuration space, the quadratic formalism is the proper one for expressing the static solutions as dynamical ones. In the first case the dynamical solution that one wants to render static has its own sense of evolution. It is then unnecessary to consider a Hamiltonian constraint with a two sheet structure in the process of parametrization. On the contrary, in the second case there is not a preferred sense of evolution for the "temporalization" of the static trajectory. Given that all the canonical variables are true degrees of freedom, there is not a "true" time disguised among them. This absence of a true time means that there is not a preferred sense of evolution for unfolding the trajectory. The Hamiltonian constraint must include then the two possible senses of evolution, which means that it must be quadratic in all its momenta.

In a quantum mechanical context these two operations assume very simple forms. Given a particular solution $\Psi_+ (q^\mu, q^0)$ to the Schrödinger equation

$$i\frac{\partial}{\partial q^0}\Psi_+ \left(q^\mu, q^0\right) = \hat{h}\Psi_+ \left(q^\mu, q^0\right) \tag{68}$$

corresponding to the sheet $p_0 + h = 0$ ($t = q^0$), its motion-reversed solution is given by

$$\Psi_+^{mr} \left(q^\mu, q^0\right) = T\Psi_+ \left(q^\mu, -q^0\right) \tag{69}$$

where T is an antiunitary operator which, in coordinate representation, is equal to the complex conjugation operator $T\Psi (q) = \Psi^\star (q)$. We could say that if the T operation changes the sense of all variables $q^i = \{q^\mu, q^0\}$, the substitution $q^0 \to -q^0$ cancels this change *only* for the variable q^0. It is important to remark that this motion reversed solution Ψ_+^{mr} is a solution of the same Schrödinger equation (68). On the contrary, the quantum version of the clock-reversal transformation is performed *only* by the action of

the antiunitary operator T (now we want to change *also* the sense of q^0):

$$\Psi_-^{cr}\left(q^\mu, q^0\right) = T\Psi_+\left(q^\mu, q^0\right). \tag{70}$$

This clock reversed solution Ψ_-^{cr} is not a solution of the Schrödinger equation (68) but a solution of the Schrödinger equation corresponding to the other sheet of the Hamiltonian constraint ($p_0 - h = 0$, $t = -q^0$):

$$-i\frac{\partial}{\partial q^0}\Psi_-\left(q^\mu, q^0\right) = \hat{h}\Psi_-\left(q^\mu, q^0\right). \tag{71}$$

It is important to remark, contrary to what is usually believed, that each sheet of the Hamiltonian constraint ($p + h = 0$ and $p - h = 0$) corresponds to positive energies solutions. What changes from one sheet to the other one is the sign of time, not the sign of the energy. Each sheet corresponds to each possible choice in the direction in which the static trajectory can be unfolded, being in both cases positive energy solutions. In fact, the stationary solutions to the Schrödinger equation (68) are

$$\Psi_+^E\left(q^\mu, q^0\right) = e^{-iEq^0}\varphi\left(q_\mu\right) = e^{-iEt}\varphi\left(q_\mu\right), \tag{72}$$

while the stationary solutions to the Schrödinger equation (71) are

$$\Psi_-^E\left(q^\mu, q^0\right) = e^{iEq^0}\varphi^*\left(q_\mu\right) = e^{-iEt}\varphi^*\left(q_\mu\right), \tag{73}$$

which shows that both sets of solutions are positive energy solutions for the two possible choices of the physical clock ($t = q^0$ or $t = -q^0$).

Following this new interpretative framework, the boundary conditions for the space of solutions of the Wheeler–DeWitt equation can be interpreted as the *symmetry breaking* of the clock-reversal invariance of a theory described by a Hamiltonian constraint with a two sheet structure. The boundary conditions proposed in Section 3.4 separate the space of solutions in those quantum states going forward in the time $t = q^0$ from those quantum states going forward in the time $t = -q^0$, being these two subspaces related by the clock-reversal operator T. The fact that the Wheeler-DeWitt operator H is a real operator has as a consequence that, given a certain solution Ψ, the associated function Ψ^* will be also

a solution, being these solutions linearly independent. The space of solutions S of the Wheeler–DeWitt equation can then be decomposed as the direct sum $S = C \oplus C^*$. In the case of the solution (37) to the Wheeler–DeWitt equation (36), the factor depending on ϕ could be expressed equivalently as a linear combination of the modified Bessel functions K_ν and I_ν or as a linear combination of the modified Bessel functions I_ν and $I_{-\nu}$. But while the functions K_ν and I_ν are not complex conjugated of each other, the functions $I_{\pm\nu}$ do satisfy the property $I_\nu = I^*_{-\nu}$. As a result, the decompositions $S = K_\nu \oplus I_\nu$ and $S = I_\nu \oplus I_{-\nu}$ are not equivalent, being the latter the correct decomposition for imposing the proposed boundary conditions. If one can find a decomposition of the space of solutions of the form $S = C \oplus C^*$, then the problem of imposing boundary conditions on this space of solutions is solved.

An important difference between the clock-reversal and the motion-reversal transformations is that a theory with a time dependent reduced Hamiltonian $h\,(t)$, even if it is not symmetric under a motion-reversal transformation, it is still symmetric under a clock-reversal transformation. This means that even if it is not possible to reduce the system by means of a time independent reduced Hamiltonian h, the space of solutions of the Wheeler-DeWitt equation must continue to be a direct sum of two subspaces related by the antiunitary operator T.

5 Conclusion

The quantization of string cosmologies is of particular interest because they allow for new points of view about the earliest stages of the universe [24, 23]. In the present work we have focused our attention in the formal aspects of the problem within the minisuperspace approximation. We shall now enumerate the central points of the whole discussion. As it is well known from quantum cosmology, the most important fact for the comprehension of the canonical structure and quantization of dilaton cosmological models is the presence of the so called *Hamiltonian constraint*. As it was explained in these notes, a theory with this kind of constraint lacks of a definition of a “true” time variable. A theory with

such a canonical structure can not be directly interpreted in terms of an evolution in time of its degrees of freedom. In order to "temporalize" the theory it is necessary to select a *physical clock*, i.e., a monotonously increasing variable which could serve as a time parameter for measuring the evolution of the rest of the canonical variables. The notion of a physical clock was not interpreted in terms of a recovered "real" hidden time, but in terms of a particular degree of freedom which satisfies certain properties which let us use it as a globally well defined clock (it is then not necessary to search for a physical clock *only* among the geometrical degrees of freedom). We have analyzed the consequences of this interpretation on the so called *parametrized formalism*, which is usually used for understanding the canonical structure of a theory with a Hamiltonian constraint *linear* in one of its momenta. It was then analyzed a modification of this parametrized formalism in order to better describe the *quadratic* Hamiltonian constraint of the gravitational models that we are studying.

If a physical clock can be separated, then the theory can be quantized in a straightforward manner by means of the parabolic Schrödinger equations associated to each sheet of the Hamiltonian constraint. If one does not know how to separate a physical clock, one can still quantize the theory by following the so called Dirac method for quantizing constrained theories. Following this method, the physical quantum states have to satisfy the operator version of the quadratic Hamiltonian constraint (Wheeler–DeWitt equation). The problem of this method is that, lacking of a definition of time, we do not know neither how to interpret the corresponding solutions in terms of a conserved positive-definite probabilities, nor how to impose boundary conditions on the space of solutions of the Wheeler–DeWitt equation. In these notes we have studied certain models which can be quantized by following both methods (Schrödinger and Wheeler–DeWitt quantizations). The important notion of a *quantum canonically transformation* was introduced in order to relate the solutions obtained by both methods. This models are of great utility in order to propose general boundary conditions for those cases for which we do not know how to reduce the model by selecting a physical clock.

Finally we have discussed the temporal symmetries which are characteristic of the

theories with a quadratic Hamiltonian constraint. Besides the *motion reversal transformation* which appears naturally in ordinary (non constrained) classical and quantum systems, we have described the so called *clock reversal transformation*. This symmetry arises as a consequence of the existence of a Hamiltonian constraint with a two sheet structure, being this symmetry associated to the passage from one sheet to the other one. We have conjectured that this symmetry could play an important role for the clarification of the boundary conditions to be imposed on the space of solutions of the Wheeler–DeWitt equation.

6 Appendix A

The condition for a function $t(q^i, p_i)$ to be a global phase time (that its Poisson bracket with the Hamiltonian constraint is positive definite) can be understood as follows. If we define the Hamiltonian vector field

$$\mathrm{H}^A = (\mathrm{H}^q, \mathrm{H}^p) = \left(\frac{\partial \mathcal{H}}{\partial p}, -\frac{\partial \mathcal{H}}{\partial q} \right), \tag{74}$$

then the condition

$$[t, \mathcal{H}] > 0$$

is equivalent to

$$\mathrm{H}^A \frac{\partial t}{\partial x^A} > 0,$$

with $x^A = (q^i, p_i)$. This means that $t(q^i, p_i)$ monotonically increases along any dynamical trajectory. Each surface $t =$ *constant* in the phase space is then crossed only once by any dynamical trajectory (so that the field lines of the Hamiltonian vector field are open). The successive states of the system can then be parametrized by $t(q^i, p_i)$.

If we explicitly write the constraint, the condition for the existence of a global time which depends only on the coordinates (*intrinsic time*) reads

$$[t(q^i), \mathcal{H}] = [t(q^i), G^{ik} p_i p_k] = 2 \frac{\partial t}{\partial q^i} G^{ik} p_k > 0. \tag{75}$$

If the supermetric has a diagonal form and one of the momenta vanishes at a given point of phase space, then no function depending only on its conjugated coordinate can

be a global phase time. For a constraint whose potential can be zero for finite values of the coordinates, the momenta p_k can be all equal to zero at a given point, and $[t(q^i), \mathcal{H}]$ can vanish. Hence an *intrinsic time* $t(q^i)$ can be identified only if the potential in the constraint has a definite sign. In the most general case a global phase time should be a function which depends also on the canonical momenta; in this case it is said that the system has an *extrinsic time* $t(q^i, p_i)$, because the momenta are related to the extrinsic curvature (see, for example [6]).

It must be noted that scaling a given constraint by a positive function does not affect the definition of time, that is, if $[t, \mathcal{H}] > 0$, then for $H = F(q)\mathcal{H}$, $F(q) > 0$ we also have $[t, H] > 0$ (we have used this property many times in these notes). From the point of view of quantization, however, things are less trivial, and the question arises about the validity of scaling the Hamiltonian. In this sense, it is important to recall that it can be shown that an operator ordering exists such that both constraints H and $\mathcal{H}$ are equivalent at the quantum level. Let us consider a generic constraint

$$\mathcal{H} = e^{bq_1}\left(-p_1^2 + p_2^2 + \zeta e^{aq_1 + cq_2}\right) = 0,$$

which contains an ambiguity associated to the fact that the most general form of the first term should be written

$$-e^{Aq_1} p_1 e^{(b-A-C)q_1} p_1 e^{Cq_1}$$

(so that A and C parametrize all possible operator orderings). It is simple to verify that the constraint with the most general ordering differs from that with the trivial ordering in two terms, one linear and one quadratic in $\hbar$, and that these terms vanish with the choice $C = b - A = 0$. Therefore, the Wheeler–DeWitt equation resulting from the scaled constraint $H = e^{-bq_1}\mathcal{H} = 0$ is right in the sense that it corresponds to a possible ordering of the original constraint $\mathcal{H}$.

7 Appendix B

Start from the parametrized action

$$S[q^i, p_i, N] = \int_{\tau_1}^{\tau_2} \left(p_i \frac{dq^i}{d\tau} - N\mathcal{H} \right) d\tau \tag{76}$$

where $\mathcal{H} = 0$ on the constraint surface. Consider the $\tau-$independent Hamilton–Jacobi equation

$$G^{ij} \frac{\partial W}{\partial q^i} \frac{\partial W}{\partial q^j} + V(q) = E \tag{77}$$

which results by substituting $p_i = \partial W / \partial q^i$ in the Hamiltonian. A complete solution $W(q^i, \alpha_\mu, E)$ (see for example [36]) obtained by matching the integration constants (α_μ, E) to the new momenta $(\overline{P}_\mu, \overline{P}_0)$ generates a canonical transformation

$$p_i = \frac{\partial W}{\partial q^i}, \qquad \overline{Q}^i = \frac{\partial W}{\partial \overline{P}_i}, \qquad \overline{K} = N\overline{P}_0 = N\mathcal{H} \tag{78}$$

which identifies the constraint $\mathcal{H}$ with the new momentum $\overline{P}_0$. The variables $(\overline{Q}^\mu, \overline{P}_\mu)$ are conserved observables because $[\overline{Q}^\mu, \mathcal{H}] = [\overline{P}_\mu, \mathcal{H}] = 0$, so that they are not suitable for characterizing the dynamical evolution. A second transformation generated by the function

$$F = P_0 \overline{Q}^0 + f(\overline{Q}^\mu, P_\mu, \tau) \tag{79}$$

gives

$$\overline{P}_0 = P_0 \qquad \overline{P}_\mu = \frac{\partial f}{\partial \overline{Q}^\mu} \qquad Q^0 = \overline{Q}^0 \qquad Q^\mu = \frac{\partial f}{\partial P_\mu} \tag{80}$$

and a new non vanishing Hamiltonian $K = NP_0 + \partial f / \partial \tau$, so that (Q^μ, P_μ) are non conserved observables because $[Q^\mu, \mathcal{H}] = [P_\mu, \mathcal{H}] = 0$ but $[Q^\mu, K] \neq 0$ and $[P_\mu, K] \neq 0$; we have, instead, that $[Q^0, \mathcal{H}] = [Q^0, P_0] = 1$, and then Q^0 can be used to fix the gauge. The transformation $(q^i, p_i) \rightarrow (Q^i, P_i)$ leads to the action

$$\mathcal{S}[Q^i, P_i, N] = \int_{\tau_1}^{\tau_2} \left(P_i \frac{dQ^i}{d\tau} - NP_0 - \frac{\partial f}{\partial \tau} \right) d\tau \tag{81}$$

which contains a linear and homogeneous constraint $P_0 = 0$ and a non zero (true) Hamiltonian $\partial f / \partial \tau$. Then we have obtained the action of an ordinary gauge system. Fixing

the gauge on this system defines a particular foliation of space-time for the associated cosmological model originally described by the parametrized action.

In terms of the original variables the action $\mathcal{S}$ reads

$$\mathcal{S}[q^i, p_i, N] = S[q^i, p_i, N] + \left[\overline{Q}^i \overline{P}_i - W + Q^\mu P_\mu - f\right]_{\tau_1}^{\tau_2} \tag{82}$$

We see that $\mathcal{S}$ and S differ only in end point terms; thus both actions yield the same dynamics. (For a review about the application to the deparametrization and also the subsequent path integral qunatization of minisuperspaces, see Ref. [43]).

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